ISRAEL JOURNAL OF MATHEMATICS **89** (1995), 71–76

CARATHÉODORY BALLS AND NORM BALLS OF THE DOMAINS $H_n = \{z \in \mathbb{C}^n : |z_1| + \cdots + |z_n| < 1\}$

BY

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ABSTRACT

In this short note we prove that the only Carathéodory balls in domains given in the title (for $n \geq 2$) which are the norm balls are the ones with the center at 0. It is a generalization of the result of B. Schwarz, who proved this theorem in case $n = 2$.

For any domain $D \subset \mathbb{C}^n$, $w, z \in D$ we define

$$c_D(w,z) = \sup\{p(\varphi(w), \varphi(z)) \colon \varphi \colon D \longrightarrow E \text{ is a holomorphic mapping}\},$$

where E is the unit disk in $\mathbb{C}$ and p is the Poincaré distance on E. c_D is called the Carathéodory pseudodistance of D. In case when D is a bounded domain, then c_D is a distance on D. We also denote

$$c_D^* \colon = \tanh c_D.$$

If D is a bounded domain, then for $z_c \in D$, $0 < r < 1$ we define the Carathéodory ball as

$$B_{c_D^*}(z_c, r) \colon = \{z \in D \colon c_D^*(z_c, z) < r\}.$$

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Received March 11, 1994

Below we shall consider the following domains

$$H_n\colon = \{z = (z_1, \ldots, z_n)\colon N_n(z) < 1\},$$

where $N_n(z)\colon = |z_1| + \cdots + |z_n|$.

We also define the following N_n-balls

$$B_{N_n}(z_N, r)\colon = \{z \in \mathbb{C}^n\colon N(z - z_N) < r\},$$

for $z_N \in \mathbb{C}^n, r > 0$.

In [S] Schwarz proved the following theorem for $n = 2$, conjectured that it is true for all $n \geq 2$, and noted that it is not true for $n = 1$.

THEOREM 1: *Let $n \geq 2$, $\overset{o}{z} \in H_n$ and $0 < r < 1$. If $B_{c^*_{H_n}}(\overset{o}{z}, r) = B_{N_n}(\overset{o}{w}, \tilde{r})$ for some $\overset{o}{w} \in \mathbb{C}^n$ and $\tilde{r} > 0$, then $\overset{o}{z} = 0$.*

The aim of this short paper is to show that the theorem remains true in the general case, as it is conjectured in [S]. As an additional remark let us also mention that the Carathéodory balls in H_n with centers at 0 are actually N_n-balls (see e.g. [S]).

Before we go into the proofs let us introduce the following notion: A holomorphic mapping $\varphi\colon E \longrightarrow D$ is a c-geodesic in D if $c_D(\varphi(\lambda_1), \varphi(\lambda_2)) = p(\lambda_1, \lambda_2)$ for any $\lambda_1, \lambda_2 \in E$.

It is well known (see [L]) that in case when D is a convex, bounded domain, then for any pairs of points $(w, z) \in D \times D$ with $w \neq z$ there is a c-geodesic $\varphi\colon E \longrightarrow D$ such that $\varphi(0) = w$ and $\varphi(c^*_D(w, z)) = z$.

Our proof is based on the above mentioned result for $n = 2$ and the characterization of c-geodesics in convex, complex ellipsoids given in [JPZ], $\mathcal{E}(p)\colon = \{|z_1|^{2p_1} + \cdots + |z_n|^{2p_n} < 1\}$, where $p = (p_1, \ldots, p_n)$, $p_j \geq \frac{1}{2}$ and $n \geq 2$, which is given by the following theorem:

THEOREM 2: (see [JPZ] also [JP]) *A bounded, holomorphic mapping $\varphi = (\varphi_1, \ldots, \varphi_n)\colon E \longrightarrow \mathbb{C}^n$ is a c-geodesic in $\mathcal{E}(p)$ if and only if*

$$\varphi_j(\lambda) = a_j \left(\frac{\lambda - \alpha_j}{1 - \bar{\alpha}_j \lambda}\right)^{r_j} \left(\frac{1 - \bar{\alpha}_j \lambda}{1 - \bar{\alpha}_0 \lambda}\right)^{1/p_j}, \tag{1}$$

or

$$\varphi_j(\lambda) = 0, \tag{2}$$

where

$$r_j \in \{0,1\}, a_j \in \mathbb{C}_*, \quad \text{for } j = 1,\dots,n, \quad \alpha_0 \in E,$$
$$\alpha_j \in E \text{ for } j \text{ such that } r_j = 1, \quad \alpha_j \in \bar{E} \text{ for } j \text{ such that } r_j = 0,$$

and the following relations hold (in the case (2) we put $\alpha_j := 0, a_j := 0, r_j := 0$*):*

$$\alpha_0 = \sum_{j=1}^{n} |a_j|^{2p_j} \alpha_j,$$
$$1 + |\alpha_0|^2 = \sum_{j=1}^{n} |a_j|^{2p_j}(1 + |\alpha_j|^2).$$

The case: for any $j = 1,\dots,n$ *the mapping* φ_j *is either of the form (2) or of the form (1) with* $r_j = 0$ *and* $\alpha_j = \alpha_0$ *is excluded. The branches of powers are taken so that* $1^{1/p_j} = 1$.

Moreover, for any pair of different points we have the uniqueness of the complex geodesics passing through these points, up to automorphisms of E.

Below we present a theorem, which is a simple consequence of Theorem 2 applied to H_n $(= \mathcal{E}(\frac{1}{2},\dots,\frac{1}{2}))$.

THEOREM 3: *Let* $z := (z_1, z_2, z_3, \dots, z_n)$ *and* $w := (w_1, w_2, z_3, \dots, z_n)$ *be distinct points in* H_n, $n \geq 2$. *Then the image of the c-geodesic through* z *and* w *is included in the set* $\mathbb{C}^2 \times \{(z_3,\dots,z_n)\}$, *and*

$$c_{H_n}(w,z) = c_{H_2}\left(\frac{(z_1,z_2)}{1 - \sum_{k=3}^{n}|z_k|}, \frac{(w_1,w_2)}{1 - \sum_{k=3}^{n}|z_k|}\right). \tag{3}$$

Proof of Theorem 3: Let us take a c-geodesic

$$\phi \colon E \longrightarrow H_2,$$

such that

$$\phi(0) = \frac{(z_1,z_2)}{1 - \sum_{k=3}^{n}|z_k|}, \quad \phi(t) = \frac{(w_1,w_2)}{1 - \sum_{k=3}^{n}|z_k|},$$

where $t > 0$. Then the numbers $a_1, a_2, \alpha_0, \alpha_1, \alpha_2$ in Theorem 2 are uniquely determined and satisfy among others the relations

$$\alpha_0 = |a_1|\alpha_1 + |a_2|\alpha_2, \tag{4}$$

$$1 + |\alpha_0|^2 = |a_1|(1 + |\alpha_1|^2) + |a_2|(1 + |\alpha_2|^2). \tag{5}$$

Let us define the holomorphic mapping

$$\tilde{\phi} := \left(\left(1-\sum_{k=3}^{n}|z_k|\right)\phi, z_3,\dots,z_n\right).$$

Since $\phi(E)\subset H_2$, we have also $\tilde{\phi}(E)\subset H_n$.

Let us also put

$$\tilde{a}_j := \begin{cases}(1-\sum_{k=3}^{n}|z_k|)\,a_j, & \text{for } j=1,2,\\ z_j, & \text{for } j=3,\dots,n;\end{cases}$$

$$\tilde{\alpha}_j := \begin{cases}\alpha_j, & \text{for } j=1,2,\\ \alpha_0, & \text{if } j=0 \text{ or } j=3,\dots,n.\end{cases}$$

Remark that in view of (4)

$$\begin{aligned}\sum_{j=1}^{n}|\tilde{a}_j|\tilde{\alpha}_j &= \left(1-\sum_{k=3}^{n}|z_k|\right)(|a_1|\alpha_1+|a_2|\alpha_2)+\sum_{j=3}^{n}|z_j|\alpha_0\\ &=\left(1-\sum_{k=3}^{n}|z_k|\right)\alpha_0+\alpha_0\sum_{j=3}^{n}|z_j|=\alpha_0=\tilde{\alpha}_0\end{aligned}\tag{6}$$

and in view of (5)

(7)

$$\begin{aligned}&\sum_{j=1}^{n}|\tilde{a}_j|\left(1+|\tilde{\alpha}_j|^2\right)\\ &=\left(1-\sum_{k=3}^{n}|z_k|\right)\left(|a_1|\left(1+|\alpha_1|^2\right)+|a_2|\left(1+|\alpha_2|^2\right)\right)+\sum_{j=3}^{n}|z_j|\left(1+|\alpha_0|^2\right)\\ &=\left(1-\sum_{k=3}^{n}|z_k|\right)\left(1+|\alpha_0|^2\right)+\left(1+|\alpha_0|^2\right)\sum_{j=3}^{n}|z_j|=1+|\tilde{\alpha}_0|^2.\end{aligned}$$

From the form of ϕ, the definition of $\tilde{\phi}$, (6) and (7) we easily conclude, in view of Theorem 2, that the mapping $\tilde{\phi}$ is a c-geodesic in H_n, $\tilde{\phi}(0)=z$ and $\tilde{\phi}(t)=w$. This implies (3) and completes the proof of Theorem 3. ∎

Now we come to the generalization of the result from [S]

Proof of Theorem 1 in case $n\geq 3$: Suppose that the theorem does not hold, so $\overset{o}{z}\neq 0$. Without loss of generality we may assume that

$$\overset{o}{z}_1\neq 0.\tag{8}$$

Let us define

$$\pi\colon \mathbb{C}^n \ni z \longrightarrow (z_1, z_2) \in \mathbb{C}^2,$$
$$\Lambda\colon \mathbb{C}^2 \ni z \longrightarrow \frac{1}{1-\sum_{k=3}^{n}|\overset{o}{z}_k|} z \in \mathbb{C}^2.$$

We know that

$$\text{(9)}\quad \begin{aligned} L\colon &= (\Lambda \circ \pi)\left(B_{c^*_{H_n}}(\overset{o}{z}, r) \cap (\mathbb{C}^2 \times \{(\overset{o}{z}_3, \dots, \overset{o}{z}_n)\})\right) \\ &= (\Lambda \circ \pi)\left(B_{N_n}(\overset{o}{w}, \tilde{r}) \cap (\mathbb{C}^2 \times \{(\overset{o}{z}_3, \dots, \overset{o}{z}_n)\})\right) =: R. \end{aligned}$$

Remark that

$$R = \Lambda\left(\left\{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1 - \overset{o}{w}_1| + |z_2 - \overset{o}{w}_2| < \tilde{r} - \sum_{k=3}^{n} |\overset{o}{z}_k - \overset{o}{w}_k|\right\}\right)$$

is an N_2-ball.

On the other hand, in view of Theorem 3

$$\begin{aligned} L = \Lambda\Big(\Big\{(z_1, z_2)\colon &(z_1, z_2, \overset{o}{z}_3, \dots, \overset{o}{z}_n) \in H_n \text{ and} \\ &c^*_{H_n}\left((z_1, z_2, \overset{o}{z}_3, \dots, \overset{o}{z}_n), \overset{o}{z}\right) < r\Big\}\Big) \\ = \Lambda\Bigg(\Bigg\{(z_1, z_2)\colon &|z_1| + |z_2| < 1 - \sum_{k=3}^{n} |\overset{o}{z}_k| \text{ and} \\ &c^*_{H_2}\left(\frac{(z_1, z_2)}{1-\sum_{k=3}^{n}|\overset{o}{z}_k|}, \frac{(\overset{o}{z}_1, \overset{o}{z}_2)}{1-\sum_{k=3}^{n}|\overset{o}{z}_k|}\right) < r\Bigg\}\Bigg) \\ = B_{c^*_{H_2}}&\left(\frac{(\overset{o}{z}_1, \overset{o}{z}_2)}{1-\sum_{k=3}^{n}|\overset{o}{z}_k|}, r\right). \end{aligned}$$

And now $L = R$ is an N_2-ball, which, in view of Theorem 1 in case $n = 2$ ([S]), contradicts (8). ∎

Added in proof: The same result as in this paper has been proved by U. Srebro, *Carathéodory balls and norm balls in* $H_n = \{z \in \mathbb{C}^n : \|z\|_1 < 1\}$, published in this volume.

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